

Conway's Army Percolation

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Conway's Soldiers is a beautiful mathematical puzzle invented by John Conway in 1961. It is concerned with an infinite chessboard, whose lower semiplane may be filled with pegs. Conway's result states that even in such a configuration with infinitely many pegs one cannot bring a peg to row 5 by checker moves.

In this paper we study a natural probabilistic version of Conway's puzzle: given constant $p \in [0, 1]$ fill every cell of the lower semiplane with a peg independently with probability p . Conway's proof (based on a potential function argument) shows that such a change in the model will not help with reaching line five. However, it may impact quantity $D_k(p)$, $k = 1, \dots, p$, defined as follows: consider a fixed cell C_k on line k . $D_k(p)$ is the probability that, starting from a random initial configuration generated as described above, we can bring a peg to cell C_k .

We are interested in the following questions:

- What are the functions $D_k(p)$, for $k = 1, 2, 3, 4$? Do they have points where they are not analytical?
- We conjecture that there exist exponents $\alpha_k, \beta_k > 0$ such that

$$D_k(p) = \Theta(p^{\alpha_k}) \text{ as } p \rightarrow 0$$

$$D_k(p) = 1 - \Theta((1 - p)^{\beta_k}) \text{ as } p \rightarrow 1.$$

What are the values of these exponents α_k, β_k , $k = 1, \dots, 4$?

- Is our version of Conway's army related in any way to percolation theory?

Results in a Nutshell

In this paper we obtain the following answers to the above questions:

- We obtain several rigorous upper and lower bounds on $D_k(p)$. The best bounds we provide are summarized in Figure 1. We only provide one type of lower bound. On the other hand, the best upper bounds combine (in a hybrid manner) two different approaches:
 - A direct, combinatorial, method.
 - Upper bounds based on tail inequalities.
- We complement these bounds by experiments that employ SAT solvers. These experiments provides estimates (also displayed in Figure 1) that suggest the fact that $D_k(p)$ are analytical functions of p .

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- We provide estimates on the critical exponents α_k, β_k (assuming that they exist), displayed in Table 1.
- Finally, we formally connect our version of Conway's army to a percolation game on hypergraphs. While on the two-dimensional lattice there is no percolation (since we cannot reach arbitrarily large heights), we show that when running this dynamics on a hypergraph generalization of the *Bethe lattice* there is a critical value p_c of the control parameter beyond which we **do** have percolation. We explicitly compute the value of p_c by an analysis that is a variant on the classical analysis of branching processes.

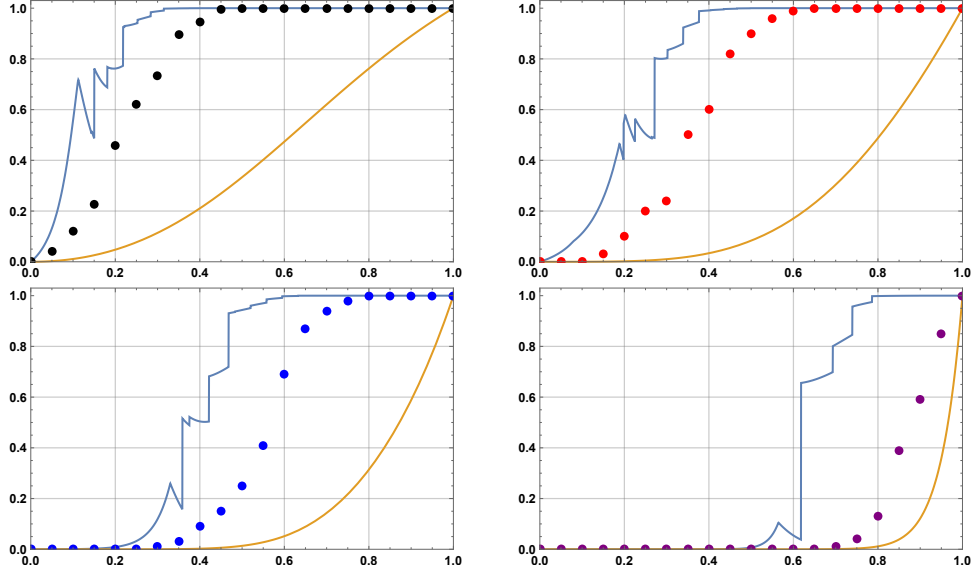


Figure 1: Best Upper Bounds, Best Lower bounds and Experimental Estimates for $D_k(p)$: (a). $k = 1$. (b). $k = 2$. (c). $k = 3$, (d). $k = 4$.

Table 1: Results and Conjectures on the Critical Exponents

Coefficient	Lower bound	Upper Bound	Conjectured Value
α_1	1	2	2
α_2	1	4	4
α_3	1	8	8
α_4	1	20	20
β_1	1	49	49
β_2	1	25	25
β_3	1	16	16
β_4	1	4	4